Math 250 Briggs 2/e

7.3 Log and exponential functions with other bases

Objectives

1) Differentiate exponential functions with bases other than e.

2) Differentiate logarithmic functions with bases other than e.

3) Integrate exponential functions with bases other than e

4) Integrate logarithmic functions with bases other than e

5) Differentiate tower functions using logarithmic differentiation

Bases other than e (logs + exponentials)

Using only integer exponents, find the following:

① 2^5 (whole number exponent)

$= 2\cdot 2\cdot 2\cdot 2\cdot 2$

$= \boxed{32}$

② $2^{1/3}$ (rational number exponent)

$= \sqrt[3]{2}$ is a number $x = \sqrt[3]{2}$

so $x^3 = 2$ Approximate by trial and error:

$x=1$: $x^3 = 1^3 = 1 < 2$ too low

$x=1.5$ $(1.5)^3 = 3.375 > 2$ too high

$x=1.2$ $(1.2)^3 = 1.728 < 2$ too low

$x=1.3$ $(1.3)^3 = 2.197 > 2$ too high

$x=1.25$ $(1.25)^3 \approx 1.953 < 2$ too low

$x=1.26$ $(1.26)^3 \approx 2.00037 > 2$ too high but not by much...

continue to desired accuracy.

$x \approx 1.259921105$

③ $2^{.41}$ (rational number exponent)

$= 2^{41/100}$

$= \sqrt[100]{2^{41}}$ is a number $x = \sqrt[100]{2^{41}}$

$x^{100} = 2^{41}$

Approximate by trial and error, as before.

ugly, but possible.

↓

④ But what about 2^π? (irrational number exponent!)

rational number approximation to π

and approximation of 2^x

coupled together could produce substantively erroneous result.

We MUST DEFINE THIS DIFFERENTLY.

Defining $f(x) = 2^x$ for any real # x

affects how we define $g(x) = \log_2(x)$ for any real # x.

Recall:

1) $\ln x = \int_1^x \frac{1}{t}\,dt$ is definition of $\ln x$ for any $x>0$

$f(x) = \ln x$ has domain $(0,\infty)$
range $(-\infty,\infty)$

2) e^x = inverse of $\ln x$ for any real # x
$g(x) = e^x$ has domain $(-\infty,\infty)$
and range $(0,\infty)$

3) Inverse properties give

$e^{\ln w} = w$ for any $w>0$ (in the domain of $\ln x$).

GOAL: Definition of b^x for any real # x, using the three facts and definitions above.

If $w = b^x$: $b^x = e^{\ln(b^x)}$
$= e^{x\cdot\ln b}$

All exponent laws still work. see notes online. → NEW DEFN $\boxed{b^x = e^{(\ln b)x}}$ we write $\ln b$ first because it is a constant

so long as $b>0$, $b\neq 1$ (a valid base) — constant
and x is a real number — variable

yes (5) Use natural log differentiation to find $\frac{dy}{dx}$ when $y = 2^x$.

$\ln y = \ln 2^x$ take natural logs of both sides

$\ln y = x\cdot\ln 2$ log property

$\ln y = \ln 2\cdot x$ recognize that $\ln 2$ is a constant.

$\frac{1}{y}\cdot\frac{dy}{dx} = \ln 2$ take derivatives of both sides, using chain rule $\frac{dy}{dx}$ for the derivatives of y.

$\frac{dy}{dx} = (\ln 2)\cdot y$ Solve for $\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = (\ln 2)\cdot 2^x}$ Substitute for y.

6. Use new definition of b^x to find $\frac{dy}{dx}$ for $y=2^x$

$y = 2^x$

$y = e^{(\ln 2)\cdot x}$ rewrite 2^x using definition

$\frac{dy}{dx} = \underbrace{e^{(\ln 2)\cdot x}}_{\text{derivative of } e^x} \cdot \underbrace{\frac{d}{dx}\left[(\ln 2)\, x\right]}_{\text{chain rule}}$ differentiate

$\frac{dy}{dx} = e^{(\ln 2)x} \cdot \ln 2$ constant multiple rule

$\boxed{\frac{dy}{dx} = (\ln 2)\cdot 2^x}$ Substitute back (defn) move constant ln 2 to front

same result as before! ☺

Math 250 Bases other than e

But there's more! We need a definition for $\log_b x$!

yes <u>Recall:</u> The change-of-base formula

$$\log_b x = \frac{\log_B x}{\log_B b} \qquad \text{for base } B > 0,\ b \neq 1$$

In particular, if our new base B is base e,

$$\log_b x = \frac{\log_e x}{\log_e b} = \frac{\ln x}{\ln b} = \frac{1}{\ln b} \cdot \ln x$$

Remember, a is a constant, so ln a is constant.

All log properties still work. See notes online.

NEW DEFN $$\boxed{\log_b x = \left[\frac{1}{\ln b}\right] \cdot \ln x}$$

So long as $b > 0$, $b \neq 1$ (b valid base) — constant
and $x > 0$ (domain of ln x) — variable

HINT for remembering change-of-base formula:

yes 7. $\log_4 7 = x$

$4^x = 7$ write in exponential form

$\ln 4^x = \ln 7$ take logs both sides

$x \cdot \ln 4 = \ln 7$ log property

$\boxed{x = \log_4 7 = \frac{\ln 7}{\ln 4}}$ divide by the constant ln 4

no 8. Use new definition of $\log_b x$ to find $\frac{dy}{dx}$ for $y = \log_2 x$.

$y = \log_2 x$

$y = \frac{1}{\ln 2} \cdot \ln x$

$\frac{dy}{dx} = \left(\frac{1}{\ln 2}\right) \cdot \frac{d}{dx}[\ln x]$ constant multiple rule

$$\boxed{\frac{dy}{dx} = \left(\frac{1}{\ln 2}\right) \cdot \frac{1}{x}}$$

Derivatives of Exponentials and Logarithms — Deriving the Formulas using ln differentiation

(9) $y = b^x$ ← x is variable, b is a constant.

$\ln y = \ln b^x$

$\ln y = x \cdot (\ln b)$ ← $(\ln b)$ is just an ugly #, a coefficient

$\ln y = (\ln b) \cdot x$

$\frac{1}{y} \cdot \frac{dy}{dx} = (\ln b)$ ← differentiate both sides, using implicit differentiation

$\frac{dy}{dx} = (\ln b) \cdot y$ ← Solve for $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = (\ln b) \cdot b^x}$$

NOTE: If $b=e$, we have the natural exponential e^x

$\frac{d}{dx}e^x = \ln e \cdot e^x = 1 \cdot e^x = e^x$

same result as before $\frac{d}{dx}e^x = e^x$.

(10) $y = b^{u(x)}$

$\ln y = \ln(b^{u(x)})$

$\ln y = u(x) \cdot (\ln b)$

$\frac{1}{y} \cdot \frac{dy}{dx} = (\ln b) \cdot u'(x)$.

$\frac{dy}{dx} = (\ln b) \cdot y \cdot u'(x)$

$$\boxed{\frac{dy}{dx} = (\ln b) \cdot b^{u(x)} \cdot u'(x)}$$

(11) $y = \log_b x$

$y = \frac{\ln x}{\ln b} = \left(\frac{1}{\ln b}\right) \cdot \ln x$

← Use defn of $\log_b x = \frac{\ln x}{\ln b}$

← $\ln b$ is a number, so $\left(\frac{1}{\ln b}\right)$ is a number, too.

$$\boxed{\frac{dy}{dx} = \left(\frac{1}{\ln b}\right) \cdot \frac{1}{x}}$$

(12) $y = \log_b(u(x))$

$y = \frac{\ln(u(x))}{\ln b} = \left(\frac{1}{\ln b}\right) \cdot \ln(u(x))$

$$\boxed{\frac{dy}{dx} = \left(\frac{1}{\ln b}\right) \cdot \frac{1}{u(x)} \cdot u'(x)}$$

Suggestion: If you forget whether to multiply or divide by $\ln b$, use log diff (above) to derive the formulas.

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FORMULAS FOR DERIVATIVE OF EXPONENTIAL base $b \neq e$

$$\frac{d}{dx}[b^x] = (\ln b) \cdot b^x$$

$$\frac{d}{dx}\left[b^{u(x)}\right] = (\ln b) \cdot b^{u(x)} \cdot u'(x)$$

mean

FORMULAS FOR INTEGRAL OF EXPONENTIAL, base other than e

$$\int (\ln b) \cdot b^x \, dx = b^x + C$$

$$\int b^x \, dx = \frac{1}{\ln b} \int \ln b \cdot b^x \, dx = \frac{1}{\ln b} \cdot b^x + C$$

$$\int b^{u(x)} \cdot u'(x) \, dx = \frac{1}{\ln b} \int \ln b \cdot b^{u(x)} \cdot u'(x) \, dx = \frac{1}{\ln b} \cdot b^{u(x)} + C$$

FORMULAS FOR DERIVATIVE OF LOGARITHM base $b \neq e$

$$\frac{d}{dx}[\log_b x] = \frac{1}{\ln b} \cdot \frac{1}{\ln x}$$

$$\frac{d}{dx}[\log_b u(x)] = \frac{1}{\ln b} \cdot \frac{1}{u(x)} \cdot u'(x)$$

mean

FORMULAS for INTEGRALS OF LOGARITHM base $a \neq e$

$$\int \frac{1}{\ln b} \cdot \frac{1}{x} \, dx = \frac{1}{\ln b} \int \frac{dx}{x} = \frac{1}{\ln b} \cdot \ln x + C = \log_b x + C$$

$$\int \frac{1}{\ln b} \cdot \frac{u'(x)}{u(x)} \, dx = \frac{1}{\ln b} \int \frac{u'(x)\,dx}{u(x)} = \frac{1}{\ln b} \cdot \ln u(x) + C = \log_b u(x) + C$$

NOTE: These derivatives are essentially those of $\ln x$ or e^x with constant $\ln b$ or $\frac{1}{\ln b}$.

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Find derivatives.

(13) yes

$y = 7^{2x-1}$

Method 1: By formula $\quad \frac{d}{dx}\left[b^{u(x)}\right] = \ln b \cdot b^{u(x)} \cdot u'(x)$

$y' = \ln 7 \cdot 7^{2x-1} \cdot 2$

$\boxed{y' = 2\ln 7 \cdot 7^{2x-1}}$

$\boxed{y' = \ln 49 \cdot 7^{2x-1}}$

$y' = 2(\ln 7)\,\frac{7^{2x}}{7}$

$y' = 2(\ln 7)\cdot\frac{(7^2)^x}{7}$

$\boxed{y' = \frac{2(\ln 7)\cdot 49^x}{7}}$

book likes

1) Bigger bases for exponentials

2) Smaller

Method 2: By log diff.

$y = 7^{2x-1}$

$\ln y = \ln 7^{2x-1}$

$\ln y = (2x-1)\cdot\ln 7$

$\frac{1}{y}\cdot\frac{dy}{dx} = \ln 7\,(2)$

$\frac{dy}{dx} = 2\ln 7 \cdot y$

$\boxed{\frac{dy}{dx} = 2\ln 7 \cdot 7^{2x-1} = \ln 49 \cdot 7^{2x-1}}$

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(14) yes

$f(x) = x^{\sqrt{2}}\cdot\sqrt{2}^{\,x}$

$f'(x) = x^{\sqrt{2}}\cdot\frac{d}{dx}\left[\sqrt{2}^{\,x}\right] + \frac{d}{dx}\left[x^{\sqrt{2}}\right]\cdot\sqrt{2}^{\,x}$ product rule

$= x^{\sqrt{2}}\cdot\underbrace{\ln\sqrt{2}\cdot\sqrt{2}^{\,x}}_{\frac{d}{dx}b^x} + \underbrace{\sqrt{2}\cdot x^{\sqrt{2}-1}}_{\text{just power rule}}\cdot\sqrt{2}^{\,x}$

$= \ln\sqrt{2}\cdot x^{\sqrt{2}}\cdot\sqrt{2}^{\,x} + x^{\sqrt{2}-1}\cdot\sqrt{2}^{\,x+1}$ exp. laws $b^n\cdot b^m = b^{n+m}$

$\boxed{f'(x) = x^{\sqrt{2}-1}\cdot\sqrt{2}^{\,x}\left[(\ln\sqrt{2})\cdot x + \sqrt{2}\right]}$ factor least powers

(15) no

$g(\alpha) = 5^{-\alpha/2}\sin 2\alpha$ Product Rule

$g'(\alpha) = 5^{-\alpha/2}\cdot\frac{d}{d\alpha}\left[\sin 2\alpha\right] + \sin 2\alpha\cdot\frac{d}{d\alpha}\left[5^{-\alpha/2}\right]$

$g'(\alpha) = 5^{-\alpha/2}\cdot\cos(2\alpha)\cdot 2 + \sin(2\alpha)\cdot\ln 5\cdot 5^{-\alpha/2}\cdot\left(-\frac{1}{2}\right)$

$\boxed{g'(\alpha) = 2\cdot\cos(2\alpha)\cdot 5^{-\alpha/2} - \frac{\ln 5}{2}\cdot\sin(2\alpha)\cdot 5^{-\alpha/2}}$

$\boxed{g'(\alpha) = \frac{5^{-\alpha/2}}{2}\left[4\cos(2\alpha) - \sin(2\alpha)\right]}$ book

Find derivatives

(16) yes

$y = x^{x-1}$ log diff

$\ln y = \ln(x^{x-1})$ Implicit diff

$\ln y = (x-1)\cdot \ln x$ product rule - RHS

$\frac{1}{y}\cdot\frac{dy}{dx} = (x-1)\cdot\frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}(x-1)$

$\frac{1}{y}\frac{dy}{dx} = (x-1)\cdot\frac{1}{x} + \ln x \cdot 1$

$\frac{dy}{dx} = y\left[\frac{x-1}{x} + \ln x\right]$

$\frac{dy}{dx} = x^{x-1}\left[\frac{x-1}{x} + \ln x\right]$

$\frac{dy}{dx} = \frac{x^{x-1}}{x}\left[x-1 + x\ln x\right]$

$$\boxed{\frac{dy}{dx} = x^{x-2}\left[x-1+x\ln x\right]}$$

(17) yes

$y = (\sin x)^{2x}$ Find equation of tangent line to the graph at $(\frac{\pi}{2}, 1)$.

check $1 = (\sin\frac{\pi}{2})^{2(\frac{\pi}{2})}$

$1 = 1^{\pi}$ ✓

$\ln y = \ln(\sin x)^{2x}$ ← log diff

$\ln y = 2x\cdot\ln(\sin x)$ ← log prop.

$\frac{1}{y}\cdot\frac{dy}{dx} = 2x\cdot\frac{d}{dx}[\ln(\sin x)] + \ln(\sin x)\cdot\frac{d}{dx}[2x]$ ← product rule

$\frac{1}{y}\cdot\frac{dy}{dx} = 2x\cdot\frac{1}{\sin x}\cdot\cos x + \ln(\sin x)\cdot 2$

$\frac{dy}{dx} = y\left[2x\cdot\cot x + 2\ln(\sin x)\right]$ ← Solve for $\frac{dy}{dx}$

$\frac{dy}{dx} = 2(\sin x)^{2x}\left[\cot x + \ln(\sin x)\right]$ ← subst for y.

$\left.\frac{dy}{dx}\right|_{(\frac{\pi}{2},1)} = 2\cdot 1\left[\cot(\frac{\pi}{2}) + \ln(\sin\frac{\pi}{2})\right]$

$= 2[0 + \ln(1)]$

$= 0.$

horizontal tangent line!

$y - 1 = 0(x - \frac{\pi}{2})$

$$\boxed{y-1=0}$$

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no (18) Find derivative of $f(x) = \log_2 \sqrt[3]{2x+1}$
(Please use log properties!)

$$f(x) = \log_2 \sqrt[3]{2x+1}$$

$$f(x) = \frac{1}{3}\log_2(2x+1)$$

Use log properties!

$$f'(x) = \frac{1}{3} \cdot \frac{1}{\ln 2} \cdot \frac{1}{2x+1} \cdot 2$$

$$f'(x) = \frac{2}{3\ln 2} \cdot \frac{1}{2x+1}$$

$$f'(x) = \frac{2}{\ln(2^3)} \cdot \frac{1}{2x+1}$$

or $\boxed{f'(x) = \frac{2}{3(2x+1)\cdot \ln 2}}$ ← book

$$f'(x) = \frac{2}{\ln 8} \cdot \frac{1}{2x+1}$$

no (19) $y = \log_2(3x)$ $x > 0$

$$y = \frac{\ln 3x}{\ln 2}$$

change-of-base formula
Note: $\ln 2$ is a constant.

$$y'(x) = \frac{1}{\ln 2} \cdot \frac{1}{3x} \cdot 3$$

chain rule of $3x$ is 3

$$\boxed{y'(x) = \frac{1}{x \ln 2}}$$

(20) $f(x) = \frac{\log_2 x}{x}$

yes

a) Find intervals of increase or decrease.
b) Find relative extrema.
c) Find intervals of concavity
d) Find inflection points.
e) Find average value on $[e, e^2]$

$$f(x) = \frac{\log_2 x}{x} = \frac{1}{x} \cdot \log_2 x = \frac{1}{x} \cdot \frac{\ln x}{\ln 2} = \frac{1}{\ln 2}\left[\frac{\ln x}{x}\right]$$

change-of-base formula

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\ln 2}\left[\frac{\ln x}{x}\right]\right)$$

$\ln 2$ is a constant, so $\frac{1}{\ln 2}$ is a constant.

$$= \frac{1}{\ln 2} \cdot \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1}{\ln 2} \cdot \frac{d}{dx}\left(x^{-1} \cdot \ln x\right)$$

quotient rule ... or rewrite $\frac{1}{x} = x^{-1}$ for product rule

$$= \frac{1}{\ln 2}\left(x^{-1} \cdot \frac{1}{x} + (-1)x^{-2} \cdot \ln x\right)$$

$$= \frac{1}{\ln 2}\left(\frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \ln x\right)$$

$$= \frac{1}{\ln 2}\left(\frac{1}{x^2} - \frac{\ln x}{x^2}\right)$$

$$= \frac{1}{\ln 2}\left(\frac{1 - \ln x}{x^2}\right)$$

$f'(x) = 0$ when numerator $1 - \ln x = 0$

$$1 = \ln x$$

$$\ln x = 1$$

$$\log_e x = 1$$

$$e^1 = x$$

$x = e$. Critical value.

$f'(x)$ undefined when denom$=0$ $\quad x=0$.

recall domain $\ln(x)$ is $x>0$

so we care only about x in domain of $f(x)$.

f' (+) (−)

0 $\quad$ e

test $x=\sqrt{e}<e$ $\qquad$ test $e^2>e$

$$f'(\sqrt{e}) \Rightarrow \frac{1-\ln\sqrt{e}}{(\sqrt{e})^2} = \frac{1-\frac{1}{2}}{e} = \frac{\frac{1}{2}}{e} > 0 \qquad f'(e^2) = \frac{1-\ln e^2}{(e^2)^2} = \frac{1-2}{e^4} = \frac{-1}{e^4} < 0$$

$\ln 2 > 0$

doesn't affect sign of result

$$\boxed{\begin{array}{ll} \text{increasing} & (0,e) \\ \text{decreasing} & (e,\infty) \\ \text{rel max} & \left(e, \frac{1}{e\ln 2}\right) \end{array}}$$

a)

b)

$$f(e) = \frac{1}{\ln 2}\left(\frac{\ln e}{e}\right) = \frac{1}{e\ln 2}$$

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left[\frac{1}{\ln 2}\left(\frac{1-\ln x}{x^2}\right)\right] = \frac{1}{\ln 2}\cdot\frac{d}{dx}\left[x^{-2}(1-\ln x)\right]$$

constant multiple

quotient rule, unless rewrite as x^{-2}

product rule

$$= \frac{1}{\ln 2}\left[x^{-2}\left(0-\frac{1}{x}\right) + (-2)x^{-3}(1-\ln x)\right]$$

$$= \frac{1}{\ln 2}\left(\frac{1}{x^2}\left(\frac{-1}{x}\right) - \frac{2}{x^3}(1-\ln x)\right)$$

$$= \frac{1}{\ln 2}\left(\frac{-1}{x^3} - \frac{2}{x^3}(1-\ln x)\right)$$

$$= \frac{1}{x^3\ln 2}\left(-1 - 2(1-\ln x)\right)$$

$$= \frac{1}{x^3\ln 2}\left(-1 - 2 + 2\ln(x)\right)$$

$$f''(x) = \frac{1}{x^3 \ln 2}\left(2\ln(x) - 3\right)$$

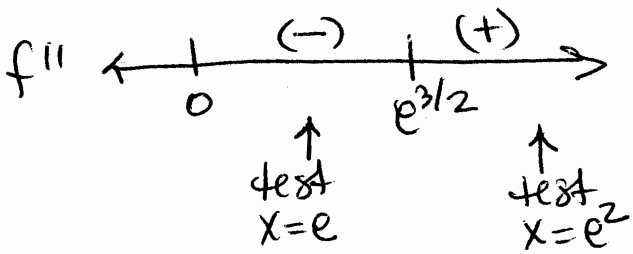
$$f''(x) = \frac{2\ln(x) - 3}{x^3 \ln 2}$$

$f''(x) = 0$ when numerator $2\ln(x) - 3 = 0$

$$2\ln(x) = 3$$
$$\ln(x) = 3/2$$
$$\log_e(x) = 3/2$$
$$e^{3/2} = x \qquad \text{equivalent exponential}$$

$f''(x)$ undefined when denom $x^3 \underbrace{\ln 2}_{\text{constant}} = 0$

$$x^3 = 0$$
$$x = 0 \qquad \text{recall domain } x > 0!$$



$$f''(e) = \frac{2\ln(e) - 3}{e^3 \ln 2} = \frac{2-3}{e^3 \ln 2} = \frac{-1}{e^3 \ln 2} < 0$$

$$f''(e^2) = \frac{2\ln(e^2) - 3}{(e^2)^3 \ln 2} = \frac{2 \cdot 2 - 3}{e^6 \ln 2} = \frac{4-3}{e^6 \ln 2} = \frac{1}{e^6 \ln 2} \quad 0$$

$$\boxed{\begin{array}{ll} \text{concave down} & (0, e^{3/2}) \\ \text{concave up} & (e^{3/2}, \infty) \\ \text{inflection point} & \left(e^{3/2}, \frac{3}{2e^{3/2}\ln 2}\right) \end{array}}$$

c)

d)

$$f(e^{3/2}) = \frac{\ln e^{3/2}}{e^{3/2} \ln 2} = \frac{3/2}{e^{3/2} \ln 2} = \frac{3}{2e^{3/2} \ln 2}$$

e) average value $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\bar{f} = \frac{1}{e^2-e}\int_e^{e^2} \underbrace{\frac{1}{\ln 2}}_{\text{constant multiple}} \cdot \frac{\ln x}{x}\,dx$$

$$= \frac{1}{e^2-e}\cdot\frac{1}{\ln 2}\int_e^{e^2} \frac{\ln x\,dx}{x}$$

$u = \ln x$

$du = \frac{dx}{x}$

$u_1 = \ln(e) = 1$

$u_2 = \ln(e^2) = 2$

$$= \frac{1}{(\ln 2)(e^2-e)}\int_1^2 u\,du$$

$$= \frac{1}{(e^2-e)\cdot\ln 2}\left[\frac{u^2}{2}\right]\Big|_1^2$$

$$= \frac{1}{(e^2-e)\cdot\ln 2}\left[\frac{2^2}{2}-\frac{1^2}{2}\right]$$

$$= \frac{1}{(e^2-e)\ln 2}\left[\frac{3}{2}\right]$$

$$= \boxed{\frac{3}{2(e^2-e)\ln 2}} \quad \text{or} \quad \boxed{\frac{3}{2e(e-1)\ln 2}} \quad \text{e)}$$

Note: The MVTI guarantees that $x=c$ in (e,e^2) exists so that $f(x)=\bar{f}$

$\approx (2.7183, 7.3891)$

$$\frac{\ln x}{x\ln 2} = \frac{3}{2e(e-1)\ln 2}$$

$$\frac{\ln x}{x} = \frac{3}{2e(e-1)}$$

but we could only approximate these value by GC

$\boxed{x \approx 5.0305 \text{ and } x \approx 1.7596}$ both in (e,e^2)

(21) Find intervals of increase, decrease
no relative extrema
intervals of concavity
inflection pts

$f(x) = x \log_2 x$ domain $x > 0$

algebra: change of base formula to natural log

$$f(x) = \frac{x \cdot \ln x}{\ln 2}$$

notice $\ln 2$ (and therefore $\frac{1}{\ln 2}$) is a constant.

$$f(x) = \frac{1}{\ln 2}\left[x \ln x\right]$$

$$f'(x) = \frac{1}{\ln 2}\left[x \cdot \frac{1}{x} + 1 \cdot \ln x\right]$$

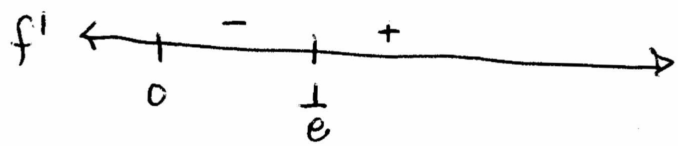
$$= \frac{1}{\ln 2}(1 + \ln x)$$

$f'(x) = 0$ $\quad 1 + \ln x = 0$

$$\ln x = -1$$

$$\log_e x = -1$$

$$e^{-1} = x$$

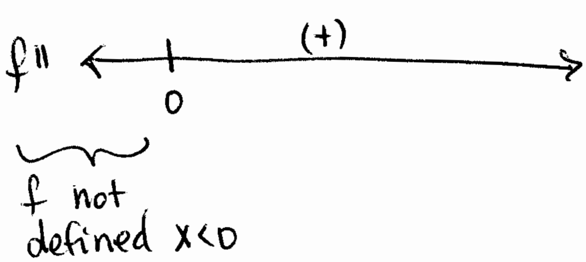


increasing $\left(\frac{1}{e}, \infty\right)$
decreasing $\left(0, \frac{1}{e}\right)$
rel min $\left(\frac{1}{e}, \frac{-1}{e\ln 2}\right)$

$$f\left(\tfrac{1}{e}\right) = \frac{1}{e} \cdot \frac{\ln\left(\frac{1}{e}\right)}{\ln 2} = \frac{1}{e} \cdot \frac{(-1)}{\ln 2}$$

$$f''(x) = \frac{1}{\ln 2}\left(0 + \frac{1}{x}\right) = \frac{1}{x \ln 2}$$

$f''(x) \neq 0$



concave up $(0, \infty)$
never concave down
no inflection points

X no (20) Find $\frac{d}{dx}(x^{\pi})$ using logarithmic differentiation.

π is constant!

$$y = x^{\pi}$$
$$\ln y = \ln x^{\pi}$$
$$\ln y = \pi \cdot \ln x$$

take natural logs of both sides and simplify using log properties

$$\frac{1}{y}\frac{dy}{dx} = \pi \cdot \frac{1}{x}$$

differentiate both sides using chain rule for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y \cdot \frac{\pi}{x}$$

isolate $\frac{dy}{dx}$

$$\frac{dy}{dx} = x^{\pi} \cdot \frac{\pi}{x}$$

subst for y

$$\boxed{\frac{dy}{dx} = \pi \cdot x^{\pi - 1}}$$

exponent law $\frac{x^a}{x^b} = x^{a-b}$

We got the same result as our power rule for integers and rationals.
So we can use the power rule for irrational exponents also!

X no (21) Find $\frac{d}{dx}(\pi^{x})$ using logarithmic differentiation.

$$y = \pi^{x}$$
$$\ln y = \ln \pi^{x}$$
$$\ln y = x \ln \pi$$
$$\frac{1}{y}\frac{dy}{dx} = \ln \pi \cdot 1$$
$$\frac{dy}{dx} = y \cdot \ln \pi$$
$$\boxed{\frac{dy}{dx} = \ln \pi \cdot \pi^{x}}$$

π is constant so $\ln \pi$ is constant!

$\frac{d}{dx}[x] = 1$

Using log diff, we can take derivatives of exponentials of any base.

Find integrals.

① $\int 5^{-x}\,dx$

$u=-x$
$du=-dx$

$= -\int 5^{-x}\cdot(-1)\,dx$

$= -\int 5^{u}\,du$

$= \frac{-1}{\ln 5}\int \ln 5\cdot 5^{u}\,du$

$= \frac{-1}{\ln 5}\cdot 5^{u}+c$

$= \boxed{\frac{-1}{\ln 5}\cdot 5^{-x}+C}$ $\quad\boxed{\frac{-5^{-x}}{\ln 5}+C}$ ← book

yes ② $\int x^3+3^{-x}\,dx$

$= \int x^3\,dx+\int 3^{-x}\,dx$

$= \boxed{\frac{x^4}{4}-\frac{3^{-x}}{\ln 3}+c}$

yes ③ $\int \frac{\ln x}{\ln 2}\,dx = \int \log_2 x\,dx$ but $\log_2 x$ is not the derivative of any function! We cannot antidiff!

yes ④ $\int \frac{\ln x}{x\ln 2}\,dx = \frac{1}{\ln 2}\int\frac{\ln x}{x}\,dx$

$u=\ln x$
$du=\frac{1}{x}\,dx$

if there's an x, we can use u-sub!

$= \frac{1}{\ln 2}\int u\,du$

$= \frac{1}{\ln 2}\cdot\frac{u^2}{2}+c = \boxed{\frac{1}{\ln 2}\cdot\frac{(\ln x)^2}{2}+c}$

Math 250

no (3) $\int (3-x)\, 7^{(3-x)^2}\, dx$

$u = (3-x)^2$

$du = 2(3-x)(-1)\, dx$

$= \frac{-1}{2} \int (3-x) \cdot 7^{(3-x)^2} \cdot (-2)\, dx$

$= \frac{-1}{2} \int 7^u\, du$

$= \frac{-1}{2} \cdot \frac{7^u}{\ln 7} + C$

$= \boxed{\frac{-1}{2} \cdot \frac{7^{(3-x)^2}}{\ln 7} + C}$

$\boxed{\frac{-1}{2\ln 7}\left[7^{(3-x)^2}\right] + C}$ ← book

yes (4) $\int_{-2}^{2} 4^{x/2}\, dx$

$u = \frac{x}{2}$

$du = \frac{1}{2} x$

$u(-2) = \frac{-2}{2} = -1$ lower

$u(2) = \frac{2}{2} = 1$ upper

$= 2\int_{-1}^{1} 4^u\, du$

$= 2 \cdot \frac{4^u}{\ln 4} \Big|_{-1}^{1}$

$= 2\left[\frac{4^1}{\ln 4} - \frac{4^{-1}}{\ln 4}\right]$

$= \frac{2}{\ln 2^2}\left[4 - \frac{1}{4}\right]$

$= \frac{2}{2\ln 2}\left[\frac{15}{4}\right]$

$= \boxed{\frac{15}{4\ln 2}}$

Proofs of Properties of Exponents using defn $a^x = e^{(\ln a)\cdot x}$

1) $a^0 = e^{(\ln a)\cdot 0} = e^0 = 1$ ✓

$\underline{\underline{1)\ a^0 = 1}}$

2) $\dfrac{a^x}{a^y} = \dfrac{e^{(\ln a)x}}{e^{(\ln a)y}} = e^{(\ln a)\cdot x - (\ln a)\cdot y}$ subtract exponents of e

$= e^{\ln a(x-y)}$ factor out ln a

$= a^{x-y}$ ✓ recognize new defn

$\underline{\underline{2)\ \dfrac{a^x}{a^y} = a^{x-y}}}$

3) $a^x \cdot a^y = e^{(\ln a)x} \cdot e^{(\ln a)y} = e^{(\ln a)x + (\ln a)y}$

$= e^{(\ln a)(x+y)}$ add exponents of e.

$= a^{x+y}$ factor out ln a

recognize new defn.

$\underline{\underline{3)\ a^x \cdot a^y = a^{x+y}}}$

4) $(a^x)^y = \left[e^{(\ln a)x}\right]^y = e^{(\ln a)\cdot x\cdot y}$ mult exp of e

$= a^{x\cdot y}$ recognize new defn

$\underline{\underline{4)\ (a^x)^y = a^{x\cdot y}}}$

PROOFS

No ↓

With these new definitions, namely

$$a^x = e^{(\ln a)\cdot x} \qquad \text{and} \qquad \log_a x = \frac{\ln x}{\ln a}$$

do our other inverse properties still work?

1) $y = a^x$ is equivalent to $x = \log_a y$.

Proof: Start with $y = a^x = e^{(\ln a)\cdot x}$

$\ln y = \ln e^{(\ln a)\cdot x}$ ← take ln

$\ln y = (\ln a)\cdot x$ ← inverse prop $\ln e^x = x$

$\frac{\ln y}{\ln a} = x$ ← divide by $\ln a$

$x = \frac{\ln y}{\ln a} = \log_a y$ ✓

Start with $x = \log_a y = \frac{\ln y}{\ln a}$

$x \cdot \ln a = \ln y$ ← mult by $\ln a$

$e^{x\cdot \ln a} = y$ ← exponentiate

$a^x = e^{\ln a \cdot x} = y$ ✓ ← inverse prop $e^{\ln a} = a$

2) $a^{\log_a x} = x$ for $x > 0$

$a^{\log_a x} = e^{\ln a \cdot \log_a x}$ ← defn a^x

$= e^{\ln a \cdot \frac{\ln x}{\ln a}}$ ← defn $\log_a x$

$= e^{\ln x}$ ← simplify

$= x$ ✓ ← prop of natural logs

3) $\log_a a^x = x$

$\log_a a^x = x \log_a a$ ← property of logs.

$= x \cdot \frac{\ln a}{\ln a}$ ← defn $\log_a a$

$= x$ ✓ ← simplify

≈25c

Properties of logs: Prove w/ new defn.

1) $\log_a(1) = 0$
2) $\log_a(xy) = \log_a(x) + \log_a(y)$
3) $\log_a x^n = n \log_a x$
4) $\log_a \frac{x}{y} = \log_a x - \log_a y$

1) $\log_a(1) = \frac{\ln 1}{\ln a} = \frac{0}{1} = 0$ ✓

2) $\log_a(xy) = \frac{\ln(xy)}{\ln(a)} = \frac{\ln(x)}{\ln a} + \frac{\ln(y)}{\ln a} = \log_a x + \log_a y$

4) $\log_a\left(\frac{x}{y}\right) \quad \frac{\ln\left(\frac{x}{y}\right)}{\ln a} = \frac{\ln(x) - \ln(y)}{\ln a} = \frac{\ln x}{\ln a} - \frac{\ln y}{\ln a} - \log_a x - \log_a y$

3) $\log_a(x^n) = \frac{\ln(x^n)}{\ln a} = \frac{n \cdot \ln(x)}{\ln a} = n \cdot \frac{\ln x}{\ln a} = n \cdot \log_a(x).$